3. L. O. Gurvich, V. Ya. Lipov, G. K. Rubin, and É. N. Sobol', Tehk. Avtomobilestroeniya, No. 8, 13-15 (1983).
4. L. O. Gurvich and É. N. Sobol', Izv. Akad. Nauk SSSR. Metally, No. 6, 154-158 (1984).
5. M. F. Ashby and K. E. Easterling, Acta Met., 32, No. 11, 1935-1948 (1984).
6. E. I. Kim, A. G. Grigor'yants, A. N. Safonov, et al., Inzh.-Fiz. Zh., 52, No. 3, 444-449 (1987).
7. V. M. Zalkin, The Study and Heat Treatment of Metals [in Russian], No. 1, 19-24 (1987).
8. J. D. Fast, "The interaction of metals and gases," in: Kinetics and the Mechanics of Reactions, Vol. 2 [Russian translation], Moscow (1975).

EXCHANGE OF HEAT IN AN ORTHOTROPIC BOUNDED CYLINDER UNDER COMBINED BOUNDARY CONDITIONS OF THE FIRST, SECOND AND THIRD KINDS

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We present the results from a study of a two-dimensional nonsteady temperature field and the heat flows in an orthotropic bounded cylinder. A method is proposed for the determination of the thermophysical properties and their ratios, as well as for the determination of the coefficient of heat exchange at the side surface.

In contemporary manufacturing, science and engineering, new materials are being utilized to an ever-increasing extent, and many of these exhibit significant nonuniformity (anisotropy). Among these materials we can include complex composition laminar structures (quasianisotropic materials).

We must recognize that the theoretical and experimental foundations on which the methods employed to study thermophysical properties (TPP) of anisotropic materials [1-5] are in need of further refinement and development [6].

It was demonstrated in [7, 8] that the solutions of two- and three-dimensional thermal conductivity problems can be employed to describe the processes of transfer in orthotropic media. For the case of heat exchange in a medium of constant temperature these solutions were made part of the method [1] for the determination of the coefficients of thermal diffusivity in solid materials. The method calls for the fabrication of two (or three) specimens of various sizes, these subsequently subjected to testing in accordance with a regular regime method. As an example of an integrated study of TPP we can point to a method based on the one-dimensional solution of the thermal-conductivity problem for a specific flow of heat [3]. The investigation in this case is carried out by specifying a recorded thermal flow, initially in one, and then in another, mutually perpendicular direction. It should be pointed out that in [1-4] no analysis is undertaken with regard to the dynamics of the development of temperature fields and flows of heat ascribed to anisotropy.

It is the purpose of the present study to develop a method for the determination of the TPP of materials, using specimens in the shape of an orthotropic bounded cylinder and to analyze the unique features encountered in the development within that cylinder of temperature fields and heat flows.

Formulation of the Problem. A bounded orthotropic cylinder of height h and diameter 2R with TPP λ_r, a_r, λ_z, a_z is given. The initial temperature of the object is T_0 = const. At some instant of time a constant flow of heat with a specific density of q_0 is applied to one of the ends of the cylinder (the coordinate origin is located in the active plane of the source), while a constant temperature T_c, different from the initial temperature, is maintained at the other end of the cylinder. At the side surface of the cylinder we note that

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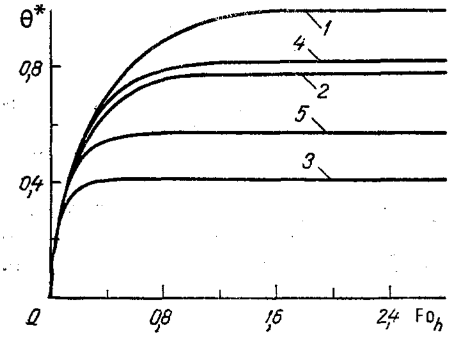

Fig. 1. The function $\Phi^*(0, 0, \tau) = f(Fo_h)$ for $Bi_R = 10$: 1) $K = 0.2$, $K_a = 1$; 2) 0.2 and 10; 3) 0.2 and 50; 4) 0.6 and 1; 5) $K = 1$, $K_a = 1$.

there takes place, based on Newton's Law, a convective exchange of heat with a medium exhibiting the initial temperature. We have to find the temperature field at any point on the cylinder for any instant of time $\tau > 0$.

The mathematical formulation of the problem is as follows: we have to solve a differential equation of the following form:

$$a_r\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right]+a_z\frac{\partial^2 T}{\partial z^2}=\frac{\partial T}{\partial \tau}\quad(\tau>0) \tag{1}$$

under the boundary conditions

$$T(r,\ z,\ 0)=T_0=\text{const}; \tag{2}$$

$$-\lambda_z\frac{\partial T(r,\ 0,\ \tau)}{\partial z}=q_0; \tag{3}$$

$$T(r,\ h,\ \tau)=T_c; \tag{4}$$

$$-\lambda_r\frac{\partial T(R,\ z,\ \tau)}{\partial r}=\alpha\,[T(R,\ z,\ \tau)-T_0]. \tag{5}$$

The solution of Eq. (1) under boundary conditions (2)-(5), obtained by means of the methods of infinite integral Laplace transforms and the finite integral Hankel transform, has the form:

$$\Theta(r,\ z,\ \text{Fo}_h)=\frac{q_0 h}{\lambda_z}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2A_m J_0\left(\delta_m\frac{r}{R}\right)\cos\left(\mu_n\frac{z}{h}\right)}{\mu_n^2+\delta_m^2 K_a K^2}\times$$
$$\times\left[1-\exp\left[-(\mu_n^2+\delta_m^2K_aK^2)\,\text{Fo}_h\right]\right]+(T_c-T_0)\times$$
$$\times\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2A_m J_0\left(\delta_m\frac{r}{R}\right)\mu_n\cos\left(\mu_n\frac{z}{h}\right)}{\mu_n^2+\delta_m^2K_aK^2}\left[1-\exp\left[-(\mu_n^2+\delta_m^2K_aK^2)\text{Fo}_h\right]\right], \tag{6}$$

where

$$A_m=\frac{2J_1(\delta_m)}{\delta_m\,[J_0^2(\delta_m)+J_1^2(\delta_m)]};$$

δ_m are the roots of the characteristic equation

$$\frac{J_0(\delta_m)}{J_1(\delta_m)}=\frac{\delta_m}{\text{Bi}_R}; \tag{7}$$

μ_n are the roots of the equation

$$\cos\mu_n=0, \tag{8}$$

i.e., $\mu_n = [(2n - 1)\pi/2]$.

The solution for (6) can be written in another form that is more convenient for the purposes of this study:

$$\Theta(r,\ z,\ \text{Fo}_h)=\frac{q_0 h}{\lambda_z}\left\{\sum_{m=1}^{\infty}\frac{A_m J_0\left(\delta_m\frac{r}{R}\right)\text{sh}\left[\delta_m\sqrt{K_a}\,K\left(1-\frac{z}{h}\right)\right]}{\delta_m\sqrt{K_a}\,K\,\text{ch}\,(\delta_m\sqrt{K_a}\,K)}-\right.$$

TABLE 1. Specific Flows in the Directions of the z and r Axes in the Steady-State Regime at Various Points of the Bounded Orthotropic Cylinder, for Various K_a ($K = 0.6$, $Bi_R = 10$)

K_a	$\frac{z}{h}$	$\frac{r}{R}=0$		$\frac{r}{R}=0{,}2$		$\frac{r}{R}=0{,}4$		$\frac{r}{R}=0{,}6$		$\frac{r}{R}=0{,}8$		$\frac{r}{R}=1$	
		$\frac{q_z}{q_0}$	$\frac{q_r}{q_0}$	$\frac{q_z}{q_0}$	$\frac{q_r}{q_0}$	$\frac{q_z}{q_0}$	$\frac{q_r}{q_0}$	$\frac{q_z}{q_0}$	$\frac{q_r}{q_0}$	$\frac{q_z}{q_0}$	$\frac{q_r}{q_0}$	$\frac{q_z}{q_0}$	$\frac{q_r}{q_0}$
0,001	0	1	0	1	0	1	0	1	0	1	0	1	0,005
	0,2	0,997	0	0,997	0	0,996	0	0,996	0	0,993	0	0,893	0,004
	0,4	0,994	0	0,994	0	0,993	0	0,991	0	0,986	0	0,839	0,003
	0,6	0,991	0	0,990	0	0,989	0	0,987	0	0,980	0	0,799	0,002
	0,8	0,988	0	0,987	0	0,986	0	0,983	0	0,973	0	0,769	0,001
	1	0,985	0	0,984	0	0,982	0	0,978	0	0,966	0	0,745	0
0,01	0	1	0	1	0	1	0	1	0	1	0	1	0.045
	0,2	0,990	0	0,990	0	0,989	0	0,986	0	0,977	0	0,748	0,032
	0,4	0,980	0	0,980	0	0,977	0	0,972	0	0,954	0,001	0,641	0,022
	0,6	0,971	0	0,970	0	0,966	0	0,958	0	0,931	0,001	0,574	0,014
	0,8	0,961	0	0,960	0	0,955	0	0,944	0	0,908	0	0,528	0,006
	1	0,951	0	0,949	0	0,944	0	0,930	0	0,885	0	0,496	0
0,1	0	1	0	1	0	1	0,002	1	0,010	1	0,045	1	0,303
	0,2	0,968	0	0,966	0,001	0,961	0,004	0,943	0,013	0,882	0,048	0,502	0,183
	0,4	0,935	0	0,932	0,001	0,920	0,004	0,885	0,013	0,774	0,042	0,358	0,113
	0,6	0,901	0	0,896	0,001	0,878	0,004	0,829	0,010	0,685	0,030	0,285	0,065
	0,8	0,868	0	0,862	0,001	0,838	0,002	0,777	0,006	0,617	0,015	0,243	0,029
	1	0,835	0	0,829	0	0,801	0	0,732	0	0,566	0	0,217	0
1	0	1	0	1	0,080	1	0,179	1	0,323	1	0,574	1	1,469
	0,2	0,807	0	0,798	0,078	0,767	0,170	0,702	0,292	0,562	0,461	0,230	0,623
	0,4	0,630	0	0,616	0,061	0,569	0,128	0,481	0,205	0,383	0,283	0,118	0,319
	0,6	0,488	0	0,473	0,039	0,425	0,079	0,341	0,119	0,220	0,151	0,075	0,159
	0,8	0,386	0	0,372	0,017	0,328	0,035	0,256	0,050	0,161	0,061	0,054	0,062
	1	0,318	0	0,306	0	0,268	0	0,207	0	0,128	0	0,042	0
100	0	1	0	1	1,239	1	2,610	1	4,331	1	6,980	1	15,902
	0,2	0,020	0	0,019	0,042	0,016	0,079	0,012	0,105	0,007	0,116	0,002	0,112
	0,4	0	0	0	0	0	0,001	0	0,001	0	0,001	0	0,001
	0,6	0	0	0	0	0	0	0	0	0	0	0	0
	0,8	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0	0	0	0
500	0	1	0	1	2,771	1	5,835	1	9,684	1	15,608	1	35,589
	0,2	0	0	0	0	0	0,001	0	0,001	0	0,001	0	0,001
	0,4	0	0	0	0	0	0	0	0	0	0	0	0
	0,6	0	0	0	0	0	0	0	0	0	0	0	0
	0,8	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0	0	0	0

$$-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2A_mJ_0\left(\delta_m\frac{r}{R}\right)\cos\left(\mu_n\frac{z}{h}\right)}{\mu_n^2+\delta_m^2K_aK^2}\times$$

$$\times\exp\left[-(\mu_n^2+\delta_m^2K_aK^2)\,\mathrm{Fo}_h\right]\Big\}+(T_c-T_0)\left\{\sum_{m=1}^{\infty}A_mJ_0\left(\delta_m\frac{r}{R}\right)\times\right.$$

$$\times\frac{\mathrm{ch}\left[\delta_m\sqrt{K_a}K\frac{z}{h}\right]}{\mathrm{ch}\left[\delta_m\sqrt{K_a}\;K\right]}+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n+1}A_mJ_0\left(\delta_m\frac{r}{R}\right)\times$$

$$\left.\times\frac{\mu_n\cos\left(\mu_n\frac{z}{h}\right)}{\mu_n^2+\delta_m^2K_aK^2}\exp\left[-(\mu_n^2+\delta_m^2K_aK^2)\mathrm{Fo}_h\right]\right\},\tag{9}$$

since [10]

$$\sum_{m=0}^{\infty}\frac{\cos\left[(2m+1)\,x\right]}{(2m+1)^2+C^2}=\frac{\pi}{4C}\frac{\mathrm{sh}\left[C\left(\frac{\pi}{2}-x\right)\right]}{\mathrm{ch}\left(\frac{\pi}{2}C\right)};$$

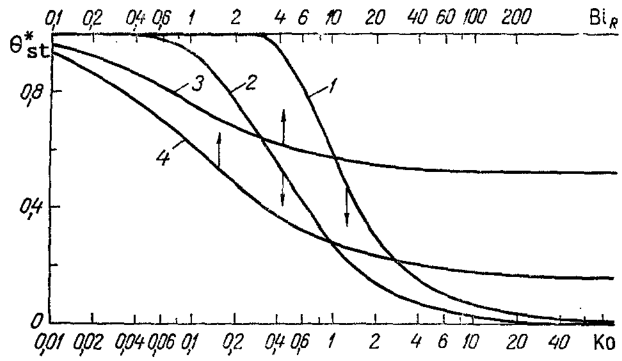

Fig. 2

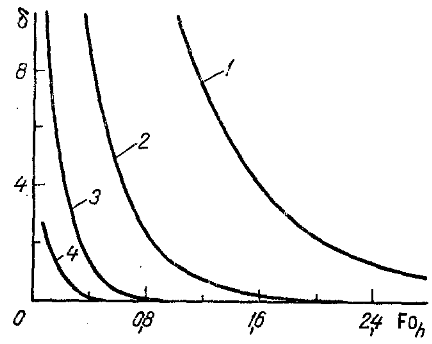

Fig. 3

Fig. 2. The function $\Phi_{st}{}^* = f(\text{Ko}, \text{Bi}_R)$ for the points r = z = 0 (curves 1, 3) and r = 0.9 R, z = 0 (curves 2, 4): 1 and 2) $\text{Bi}_R = 10$; for curves 3 and 4) Ko = 1.

Fig. 3. The function of the relative contribution of the first term in series (22) to the sum of the entire series (in percent) for $\text{Bi}_R = 10$; 1) Ko = 0.27; 2) 0.4; 3) 0.6; 4) Ko = 0.8. δ, %.

$$\sum_{m=0}^{\infty}(-1)^m \frac{(2m+1)\cos[(2m+1)x]}{(2m+1)^2+C^2} = \frac{\pi}{4}\frac{\operatorname{ch}(Cx)}{\operatorname{ch}\left(\frac{\pi}{2}C\right)}.$$

In the particular case in which the material being investigated is isotropic, i.e., $K_a = 1$, the solution of (9) coincides with the solution presented in [11].

Analysis of (9) shows that the entire heat-exchange process can be broken down into three stages: one that is purely nonstationary, a regular stage, and the steady-state stage with the steady distribution of temperature dependent on the Kirpichev criterion and the excess temperature $T_C - T_0$.

One of the practical aspect of applying solution (9) is the possibility of calculating the temperature fields and thermal flows in real orthotropic objects of cylindrical shape. However, the fundamental goal of the subsequent analysis is the application of expression (9) to determined TPP.

The initial formulation of the problem and the solution for (9) makes it possible for us to recommend various methods of integrated research into the TPP and the coefficient of heat exchange. Let us consider some of these.

Solution (9) represents the superposition of two thermal-conductivity problems corresponding to the following particular cases:

1) $q_0 = 0,\ T_C \neq T_0$;

2) $q_0 \neq 0,\ T_C = T_0$.

Let us limit ourselves to an examination of the second case, writing a simplified version of solution (9) in criterial form:

$$\Theta^*(r,\ z,\ \text{Fo}_h) = \sum_{m=1}^{\infty} A_m J_0\left(\delta_m \frac{r}{R}\right)\frac{\operatorname{sh}\left[\delta_m\sqrt{K_a}\,K\left(1-\frac{z}{h}\right)\right]}{\delta_m\sqrt{K_a}\,K\operatorname{ch}(\delta_m\sqrt{K_a}\,K)} - \tag{10}$$

$$-2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_m J_0\left(\delta_m\frac{r}{R}\right)\frac{\cos\left(\mu_n\frac{z}{h}\right)}{\mu_n^2+\delta_m^2 K_a K^2}\exp[-(\mu_n^2+\delta_m^2 K_a K^2)\text{Fo}_h] = \Theta^*_{st}(r,\ z) + \Theta^*_{nonst}(r,\ z,\ \text{Fo}_h).$$

If we assume that $K_a = 1$, we obtain the solution, given in [9], which as K → 0 with consideration of the following relationships from [10]

TABLE 2. The Ratio $N_1 = [\Phi_{st}^*(r_2, 0)/\Phi_{st}^*(r_1, 0)]$ as a Function of Bi_R for Various K

Bi_R	K 0,15	K 0,2	K 0,25	Bi_R	K 0,15	K 0,2	K 0,25
0,1	0,993	0,991	0,988	20	0,795	0,707	0,634
0,5	0,981	0,972	0,960	25	0,774	0,686	0,612
1	0,968	0,948	0,927	30	0,770	0,678	0,604
2	0,951	0,916	0,880	40	0,756	0,662	0,387
3	0,925	0,880	0,837	50	0,741	0,648	0,576
5	0,894	0,835	0,779	60	0,734	0,640	0,566
7	0,869	0,802	0,740	70	0,729	0,634	0,560
10	0,846	0,769	0,701	80	0,728	0,633	0,559
13	0,826	0,744	0,673	90	0,726	0,630	0,555
15	0,816	0,731	0,659	100	0,717	0,623	0,552
17	0,807	0,720	0,648	∞	0,689	0,594	0,521

$$\sum_{n=1}^{\infty} \frac{2\cos\left(\mu_n \frac{z}{h}\right)}{\mu_n^2} = 1 - \frac{z}{h},$$

$$\sum_{m=1}^{\infty} \frac{2J_0\left(\delta_m \frac{r}{R}\right) J_1(\delta_m)}{\delta_m [J_0^2(\delta_m) + J_1^2(\delta_m)]} = 1$$

changes into the familiar one-dimensional solution [9]:

$$\Theta^*(z, \mathrm{Fo}_h) = 1 - \frac{z}{h} - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\mu_n^2} \sin\left(\mu_n \frac{h-z}{h}\right) \exp(-\mu_n^2 \mathrm{Fo}_h) = \Theta_{st}^*(z) - \Theta_{nonst}^*(z, \mathrm{Fo}_h), \quad (11)$$

where the numbers μ_n are determined from the relationship (8).

It follows from an analysis of (10) that the propagation of heat in the radial direction (the appearance of two-dimensionality) at any point on the orthotropic cylinder depends not only on the conditions of heat exchange at the side surface and on the ratio of the geometric dimensions, but also on the relationship between the TPP, i.e., on the parameter $K_a = K_\lambda$. This relationship is clearly seen in the graphs shown in Fig. 1. With K = 0.2 and $K_a \leq 1$ the temperature at the center of the cylinder corresponds to the "one-dimensional" (curve 1) and may be described by relationship (11). But when $K_a > 1$, all other conditions being equal, the one-dimensionality is disrupted (curves 2, 3). With an increase in K the one-dimensionality is also violated (curves 4, 5). However, even with K, but with $K_a \ll 1$, the dependence on temperature is imposed on curve 1.

Consequently, in solution (10) we must deal with the complex $K\sqrt{K_a}$, which is naturally contained here, and which places the same role as the parameter K in the two-dimensional solution for the isotropically bounded cylinder.

We will refer to this complex $Ko = K\sqrt{K_a}$ as the criterion of temperature two-dimensionality for an orthotropic cylinder of finite dimensions. Then, on the basis of [9], we can draw the conclusion that for any Bi_R and $Ko \leq 0.2$, on the r = 0 axis in an orthotropic bounded cylinder, the excess temperature of $\Phi^*(0, z, \mathrm{Fo}_h)$, including the steady state, will be described with a rather small aurora (0.18%) by the one-dimensional solution (11). Since in the isotropic cylinder with $K \leq 0.2$ and for any Bi_R the temperature at the center (r = z = 0) corresponds to one that is one-dimensional, and objective experimental criterion of the anisotropy of the material being studied is the difference $\Phi_{st}^*(0, 0, \infty)$ from unity in the steady-state regime. This is valid for $K_a > 1$. However, K can be chosen in a manner such that Ko is smaller than 0.2, even when $K_a < 1$. The objective anisotropy criterion for bodies with $K_a < 1$ will then be equality to unity for $\Phi_{st}^*(0, 0, \infty)$ when K > 0.25.

Let us examine solution (10) and the specific flows of heat in the directions of the coordinate axes for the steady-state regime in the limit cases in which $K_a \to 0$ and $K_a \to \infty$. The flows of heat for the steady-state regime are determined from the following expression:

$$\frac{q_z(r, z)}{q_0} = \sum_{m=1}^{\infty} A_m J_0\left(\delta_m \frac{r}{R}\right) \frac{\mathrm{ch}\left[\delta_m \sqrt{K_a} K \left(1 - \frac{z}{h}\right)\right]}{\mathrm{ch}(\delta_m \sqrt{K_a} K)}; \quad (12)$$

$$\frac{q_r(r,\ z)}{q_0} = \sqrt{K_a} \sum_{m=1}^{\infty} \frac{2J_1\left(\delta_m \frac{r}{R}\right) J_1(\delta_m)\,\mathrm{sh}\left[\delta_m \sqrt{K_a}\, K\left(1-\frac{z}{h}\right)\right]}{\delta_m [J_0^2(\delta_m)+J_1^2(\delta_m)]\,\mathrm{ch}(\delta_m \sqrt{K_a}\, K)}. \tag{13}$$

Then

$$\lim_{K_a \to 0} \Theta^*(r,\ z,\ \mathrm{Fo}_h) = 1 - \frac{z}{h} - 2\sum_{n=1}^{\infty} \frac{\cos\left(\mu_n \frac{z}{h}\right)}{\mu_n^2} \exp(-\mu_n^2\, \mathrm{Fo}_h); \tag{14}$$

$$\lim_{K_a \to 0} \frac{q_z(r,\ z)}{q_0} = 1; \tag{15}$$

$$\lim_{K_a \to 0} \frac{q_r(r,\ z)}{q_0} = 0; \tag{16}$$

$$\lim_{K_a \to \infty} \Theta^*(r,\ z,\ \mathrm{Fo}_h) = 0; \tag{17}$$

$$\lim_{K_a \to \infty} \frac{q_z(r,\ z)}{q_0} = \begin{cases} 1 \text{ with } z = 0; \\ 0 \text{ with } 0 < z \leqslant h; \end{cases} \tag{18}$$

$$\lim_{K_a \to \infty} \frac{q_r(r,\ z)}{q_0} = \infty. \tag{19}$$

Through analysis of expressions (14)-(19) we can draw the conclusion that when $K_a = K_\lambda \to 0$ ($a_r \ll a_z$) for any Bi_R the temperature field over the entire volume of the orthotropic cylinder will be described by a one-dimensional solution. There is an absence of a flow of heat in the radial direction, while in the direction of the z axis the heat flow q_0 is transferred entirely from the z = 0 plane to the z = h plane, i.e., in this case the object under consideration serves as an ideal conductor of heat without any losses of heat to the side surface. We make no provision here for the quantity of heat which is expended on the actual heating of the body.

As $K_a \to \infty$ ($a_r \gg a_z$) for any $\mathrm{Bi}_R \neq 0$ the specimen being studied is not heated (the temperature over the entire volume of the cylinder is equal to the initial temperature). There is an absence of a flow of heat in the direction of the z axis when $0 < z \leq h$, while in the radial direction it is infinite, i.e., the heat is instantaneously removed in the z = 0 plane. In this case, the orthotropic cylinder is an ideal heat insulator. However, if the coefficient of heat exchange at the side surface is equal to zero, i.e., $\mathrm{Bi}_R = 0$, the characteristic equation (7) assumes the form of $J_1(\delta_m) = 0$ and the temperature field for the cylinder will be described by the one-dimensional solution (11) with TPP a_z, λ_z.

After we have analyzed expressions (10) and (17)-(19), we find that as z = 0 increases there is a reduction in the height of the inside layer of the cylinder (adjacent to the K_a ($K_a > 1$), where the excess temperature and specific heat flow q_z are different from zero. Where there is sufficiently large K_a, the height of this layer tends to zero. The radial flow of heat increases to the maximum. With K_a values sufficiently small ($K_a < 1$) the radial heat flow diminishes to zero while the flow along the z axis increases to q_0 at all points of the orthotropic cylinder. This is confirmed by the data presented in Table 1.

Let us now look at the temperature field of an orthotropic bounded cylinder in the steady-state regime. The excess relative temperature in this case will be determined from the expression

$$\Theta^*_{st}(r,\ z) = \sum_{m=1}^{\infty} A_m J_0\left(\delta_m \frac{r}{R}\right) \frac{\mathrm{sh}\left[\delta_m \mathrm{Ko}\left(1-\frac{z}{h}\right)\right]}{\delta_m \mathrm{Ko}\,\mathrm{ch}(\delta_m \mathrm{Ko})} = f(\mathrm{Ko},\ \mathrm{Bi}_R), \tag{20}$$

where

$$\mathrm{Ko} = K\sqrt{K_a}\,. \tag{21}$$

Figure 2 shows a graphic representation of the function in (20) for the points z = 0; r = 0 and z = 0; r = 0.9R.

It follows from an analysis of the graphs (curves 1 and 2) that, for a given Bi_R [in which case $\Phi_{st}^* = f(\mathrm{Ko})$] it is possible to determine the Ko criteria from the measurement results obtained on the excess temperature at two points; however, this applies only to a specific interval. Thus, at the point $r = 0$; $z = 0$ Ko can be determined in the range 0.3-40 (curve 1), while at the point $r = 0.9R$; $z = 0$ it is determined in the range from 0.04-20 (curve 2). The ratio of excess temperatures at these points yields the range of 0.04-20 for the determination of Ko. If we know the ratio of the geometric dimensions K, from formula (21) we can find the TPP ratio. Thus, for example, with $K_a = 0.2$ the range for the determination of K_a will be 0.4-10,000, while for $K = 1$ it will be 0.0016-400, i.e., by varying the magnitude of the parameter K we can shift the range for the determination of K_a. With a fixed value for the quantity Ko $[\Phi_{st}^* = \varphi(Bi_R)]$ and with $Bi_R \geq 1$ the relationship between the excess temperature and the Biot number is insignificant at specific points. However, this relationship is intensified when $Bi_R < 1$ (see curves 3 and 4). For Ko < 0.3 (Bi_R can be any number) and Ko > 10 ($Bi_R \neq 0$) the indicated relationship to temperature on the part of Bi_R is also insignificant.

Thus, in the steady-state thermal regime, we have the possibility of determining the Ko criterion for a known value of Bi_R, which can be easily found experimentally in the regular thermal regime [1].

Let us examine the nonsteady component of solution (10):

$$\Theta^*_{nonst}(r,\ z,\ Fo_h) = \Theta^*_{st}(r,\ z) - \Theta^*(r,\ z,\ Fo_h) = \\ = 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m J_0\left(\delta_m \frac{r}{R}\right)\cos\left(\mu_n \frac{z}{h}\right)}{\mu_n^2 + \delta_m^2 Ko^2}\exp[-(\mu_n^2 + \delta_m^2 Ko^2) Fo_h] = f(Ko,\ Bi_R,\ Fo_h). \tag{22}$$

The double series in (22) converges rapidly and for specific values of the Ko and Fo_h numbers we can, with a sufficient degree of accuracy, limit ourselves to the first term of this series, i.e., a regular heat-exchange regime sets in. The time for the onset (τ_p) of the regular regime depends significantly on the value of Ko: the larger this number, the smaller the time τ_p. This is well-illustrated by the graphs in Fig. 3.

Thus, in the regular thermal regime series (22) can be replaced by the first of its terms. The ratio Φ_{nonst}^* at two different points of the orthotropic cylinder (for example, $r_1 = z = 0$; $r_2 = 0.9R$, $z = 0$) at one and the same instant of time $\tau_1 > \tau_p$ will then be a function exclusively of the first root of the characteristic equation (7):

$$\frac{\Theta^*_{nonst}(r_1,\ 0,\ \tau_1)}{\Theta^*_{nonst}(r_2,\ 0,\ \tau_1)} = \frac{1}{J_0(0.9\delta_1)} = f(\delta_1). \tag{23}$$

Using expressions (23) and (7), we can determine the criteria and roots of Eq. (7). With $Bi_R = \infty$ the roots δ_m are found from the equation $J_0(\delta_m) = 0$.

In the regular thermal regime, in the presence of heat sources and sinks, for an isotropic unbounded plate, the rates of change in temperature [see (11)] are determined from the following formula [9]:

$$M_1 = \frac{\ln\dfrac{\Theta^*_{st} - \Theta^*(z,\ \tau_1)}{\Theta^*_{st} - \Theta^*(z,\ \tau_2)}}{\tau_2 - \tau_1} = \frac{a\mu_1^2}{h^2} \quad (\tau_2 > \tau_1 > \tau_r), \tag{24}$$

i.e., the rate of heating is proportional to the coefficient of thermal diffusivity. In the case of an orthotropic cylinder ($K_a \neq 1$) the rate of change in temperature

$$M_2 = \frac{\ln\dfrac{\Theta^*_{st} - \Theta^*(r,\ z,\ \tau_1)}{\Theta^*_{st} - \Theta^*(r,\ z,\ \tau_2)}}{\tau_2 - \tau_1} = (\mu_1^2 + \delta_1^2 Ko^2)\frac{a_z}{h^2} = \frac{\mu_1^2}{h^2}a_z + \frac{\delta_1^2}{R^2}a_r \tag{25}$$

is proportional to the sum of the thermal diffusivities in the direction of the coordinate axes. Then, with known δ_1 and Ko we can determine a_z and a_r.

Based on the above, we propose the following method for the determination of the TPP of an orthotropic bounded cylinder and the coefficient of heat exchange at the side surface.

1. The case of an isotropic material ($K_a = K_\lambda = 1$). We select the ratio of the geometric dimensions K such that the condition of temperature one-dimensionality is satisfied at the center of the cylinder (at the $r = 0$ axis), and this is meant to include the steady-state regime. Then, having measured the excess temperature, for example, at the points $r_1 = z = 0$ and $r_2 = 0.9R$, $z = 0$, we can determine the TPP of the body being studied. We deter-

mined the coefficient of thermal conductivity from the one-dimensional solution (11) in the steady regime, while we determined the coefficient of thermal diffusivity in the regular regime, using expression (24) [9]. The ratio of steady temperatures at the points that we have selected will be a function only of the Bi_R number:

$$N_1 = \frac{\Theta^*_{st}(r_2, 0)}{\Theta^*_{st}(r_1, 0)} = f(Bi_R). \tag{26}$$

Expression (26) can be calculated in advance and set up in the form of a table or nomogram (see Table 2). Having experimentally calculated the number N_1, we determined the value of the Bi_R from the table, and then the coefficient of heat exchange at the side surface, namely:

$$\alpha = \frac{Bi_R \lambda}{R}. \tag{27}$$

2. $K_a = K_\lambda$ can be any arbitrary value. We will measure the excess temperature at the same point as in the first case. The Bi_R number will then be found in the regular heat-exchange stage, using expressions (23) and (7). Then, using the steady regime, we determined Ko and K_a by means of (20) and (21). After this, based on the information about the regular regime from (25) we find a_z; a_r is found from the following formula:

$$a_r = K_a a_z.$$

The coefficients of thermal conductivity in the directions of the coordinate axes are equal to:

$$\lambda_z = \frac{q_0 h}{\Theta_{st}(0, 0) - \Theta(0, 0, \tau_1)} \left\{ \frac{4J_1(\delta_1)}{\delta_1 [J_0^2(\delta_1) + J_1^2(\delta_1)] (\mu_1^2 + \delta_1^2 Ko^2)} \exp\left[-(\mu_1^2 + \delta_1^2 Ko^2) \frac{a_z \tau_1}{h^2} \right] \right\} \quad (\tau_1 > \tau_r)$$

$$\lambda_r = K_a \lambda_z.$$

The heat-exchange coefficient for the side surface of the cylinder is determined from formula (27), having replaced λ by the value of λ_r.

Let us examine the case in which the excess temperatures at the indicated two points are equal to each other and not equal to zero. This means that $a_z \gg a_r$ and the determination of TPP is possible only along the z axis. However, if these excess temperatures are equal to zero, then $a_r \gg a_z$ [see (17)] and it is impossible to determine the TPP in this case. This is characteristic of any plane $0 \le z < h$.

NOTATION

$\Phi(r, z, \tau) = T(r, z, \tau) - T_0$, the excess temperature; $\Phi^*(r, z, \tau) = \Phi(r, z, \tau)/Ki\, T_0$, dimensionless and excess temperature; $Ki = q_0 h/\lambda_z T_0$, Kirpichev number; τ, τ_r, current time and time of regular thermal regime onset; r, z, instantaneous coordinates; $K = h/R$, ratio of geometric dimension for the orthotropic cylinder; λ_r, λ_z, a_r, a_z, coefficients of thermal conductivity and thermal diffusivity in the directions of the r and z axes; $K_a = K_\lambda = \lambda_r/\lambda_z = a_r/a_z$, ratio of thermophysical properties; $Ko = K\sqrt{K_a}$, orthotropic cylinder criterion of temperature two-dimensionality; $Bi_R = \alpha R/\lambda_r$, Biot number; $Fo_h = a_z \tau/h^2$, Fourier criterion; $J_0(x)$, $J_1(x)$, Bessel functions of the first kind.

LITERATURE CITED

1. G. M. Kondrat'ev, The Regular Thermal Regime [in Russian], Moscow, Leningrad (1954).
2. Ya. S. Podstrigach, V. A. Lomakin, and Yu. M. Kolyano, Thermoelasticity of Bodies with Nonuniform Structure [in Russian], Moscow (1984).
3. L. S. Domorod, L. E. Evseeva, and S. A. Tanaeva, Inzh.-Fiz. Zh., 32, No. 6, 1115-1116 (1977).
4. V. V. Vlasov, Yu. S. Shatalov, et al., Inzh.-Fiz. Zh., 33, No. 3, 479-485 (1977).
5. A. I. Zhornik and É. M. Kartashov, Izv. Vuzov. Aviat. Tekh., No. 3, 128-132 (1973).
6. É. M. Kartashov, Analytical Methods in the Theory of Thermal Conductivity in Solids [in Russian], Moscow (1985).
7. A. V. Lykov, The Theory of Heat Conduction [in Russian], Moscow (1967).
8. G. Karslou and D. Eger, Heat Conduction in Solids [Russian translation], Moscow (1964).
9. A. G. Shashkov, G. M. Volokhov, T. N. Abramenko, and V. P. Kozlov, Methods for the Determination of Thermal Conductivity and Thermal Diffusivity, A. V. Lykov (ed.) [in Russian], Moscow (1973).

10. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series. Elementary Functions [in Russian], Moscow (1981).
11. V. P. Kozlov, Two-Dimensional Axisymmetric Nonsteady Problems of Thermal Conductivity [in Russian], Minsk (1986).

SPECTRAL INFLUENCE FUNCTIONS OF BOUNDARY EFFECTS IN PROBLEMS DEALING WITH CONTROL OF THERMAL OBJECTS

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We describe a method for solving problems dealing with the optimum control of a thermal object that is based on high-speed action where limitations are imposed on the control function and on the thermal state.

The problem of optimum high-speed control of a thermal object whose dynamics are described by the following equation of heat conduction

$$\frac{\partial T(x,\ y,\ \mathrm{Fo})}{\partial\,\mathrm{Fo}} = \nabla^2 T(x,\ y,\ \mathrm{Fo}), \tag{1}$$

consists in the determination of such control boundary effects, expressed by piecewise polynomial functions

$$q_i(x,\ y,\ \mathrm{Fo}) = \sum_{j=0}^{m_i} a_{ij}(\mathrm{Fo})\,\xi^j;\ i = 1,\ 2,\ \ldots,\ n;\ j = 0,\ 1,\ \ldots,\ m_i, \tag{2}$$

which would ensure the transition of the object from a given initial thermal state T(x, y, 0) to the final state T(x, y, Fo_k) within a minimum of time, with satisfaction of the limitations imposed both on the controling action (the external limitation)

$$q_{\min} \leqslant q_i(x,\ y,\ \mathrm{Fo}) \leqslant q_{\max}, \tag{3}$$

as well as on the thermal state of the object (an internal system of limitations)

$$T(x,\ y,\ \mathrm{Fo}) \leqslant T_{\mathrm{per}}, \tag{4}$$

$$\Delta T(x,\ y,\ \mathrm{Fo}) \leqslant \Delta T_{\mathrm{per}}. \tag{5}$$

We will adopt the attainment of the maximum possible rate of temperature change in the object in conjunction with the given limitations (3)-(5) as the criteria of optimum control.

Applying an implicit finite-difference approximation to Eq. (1), for k-th instant of time we obtain

$$\nabla^2 T^{(k)}(x,\ y) - (\Delta\mathrm{Fo})^{-1} T^{(k)}(x,\ y) = -(\Delta\,\mathrm{Fo})^{-1} T^{(k-1)}(x,\ y). \tag{6}$$

If we specify the spectral component ξ^j as the controling action on the i-th segment of the object's boundary, with zero actions specified for the remaining (n − 1) segments, and if we solve system of equations (6) with the zero initial conditions T(x, y, 0) = 0, we will obtain the spectral influence function (SIF) $W_{ij}(x, y)$ [1].

Having thus determined the remaining SIF, we will represent the temperature at the observation point s for the k-th instant of time in the form

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